# DISCRETE AND EMBEDDED EIGENVALUES FOR ONE-DIMENSIONAL SCHRÖDINGER OPERATORS

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Abstract. I present an example of a discrete Schrödinger operator that shows that it is possible to have embedded singular spectrum and, at the same time, discrete eigenvalues that approach the edges of the essential spectrum (much) faster than exponentially. This settles a conjecture of Simon (in the negative). The potential is of von Neumann-Wigner type, with careful navigation around a previously identified borderline situation.

## 1. Introduction

I am interested in one-dimensional discrete Schrödinger equations,

$$u(n+1)+u(n-1)+V(n)u(n)=Eu(n), \tag{1.1}$$

and the associated self-adjoint operators

$$(Hu)(n)=\begin{cases} u(n+1)+u(n-1)+V(n)u(n) & (n\ge 2)\\ u(2)+V(1)u(1) & (n=1)\end{cases}$$

on $\ell_2(\mathbb{N})$. We could also consider whole line operators (on $\ell_2(\mathbb{Z})$), and for the purposes of this paper, that would actually make very little difference.

Recent work has shown that there are fascinating and unexpected relations between the discrete and essential spectrum of $H$. If $V\equiv 0$, then $\sigma_{ac}(H)=[-2,2]$, $\sigma_{sing}(H)=\emptyset$. It turns out that perturbations that substantially change the character of the spectrum on $[-2,2]$ must also introduce new spectrum outside this interval. Indeed, Damanik and Killip [3] proved the spectacular result that if $\sigma\setminus[-2,2]$ is finite, then $[-2,2]\subset\sigma$ and the spectrum continues to be purely absolutely continuous on $[-2,2]$. The situation when $\sigma\setminus[-2,2]$ is a possibly infinite set of discrete eigenvalues, with $\pm 2$ being the only possible accumulation points, was subsequently investigated by Damanik and myself [5] (honesty demands that I point out that we actually treated the analogous problems in the continuous setting).

A natural question, which was not addressed in [3, 5], concerns the minimal assumptions that will still imply that the spectrum is purely absolutely continuous on $[-2,2]$ in these situations. Put differently: *How fast can the discrete eigenvalues approach the edges of the essential spectrum $\sigma_{ess}=[-2,2]$ if the essential spectrum is* not *purely absolutely continuous? Is there in fact any bound on this rate of convergence?*

So we assume that $\sigma\setminus[-2,2]=\{E_n\}$, and we introduce

$$d_n\equiv \text{dist }(E_n,[-2,2]).$$

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We also assume that $\{E_n\}$ is infinite and $d_n \to 0$. It would in fact be natural (but not really necessary) to arrange the eigenvalues so that $d_1 \geq d_2 \geq \ldots$.

The hunt for examples with some singular spectrum on $[-2, 2]$, but rapidly decreasing $d_n$'s was opened by Damanik, Killip, and Simon in [4]. This paper has an example where $d_n \lesssim e^{-cn}$ and $0 \in \sigma_{pp}$. Based on this, Simon conjectured [8, Sect. 13.5] that the slightly stronger condition $\sum 1/\ln d_n^{-1} < \infty$ might suffice to conclude that the spectrum is purely absolutely continuous on $[-2, 2]$. The purpose of this paper is to improve on the counterexample from [4]; this will also show that the above conjecture needs to be adjusted. More precisely, we will prove:

**Theorem 1.1.** *There exists a potential $V$ so that: (1) For $E = 0$, the Schrödinger equation* (1.1) *has an $\ell_2$ solution $u$.*
*(2) $d_n \leq e^{-cn^2}$ for some $c > 0$.*

Such a potential $V$ is in fact explicitly given by

$$V(n) = \frac{(-1)^n}{n}\left(1 + \frac{2}{\ln n}\right) \qquad (n \geq 3); \tag{1.2}$$

here 2 could be replaced by any other constant $c > 1$. If a complete definition of $V$ is desired, we can put $V(1) = V(2) = 0$. However, the behavior of $V$ on finite sets is quite irrelevant for what we do here. Note also in this context that by adjusting $V(1)$, say, we can achieve that $0 \in \sigma_{pp}$.

To motivate (1.2), let us for a moment consider the simpler potential $V(n) = g(-1)^n/n$. This is basically a discrete variant of the classical von Neumann-Wigner potential [10]. The values $g = \pm 1$ for the coupling constant are critical in two senses: First of all, there exists a square summable solution to the Schrödinger equation (1.1) at energy $E = 0$ if and only if $|g| > 1$. Second, if $|g| \leq 1$, then the operator $H$ has no spectrum outside $[-2, 2]$ [2, Proposition 5.9]. On the other hand, if $|g| > 1$, there must be infinitely many eigenvalues $E_n$ with $|E_n| > 2$ by the result from [3] discussed above. A rather detailed analysis is possible, and $V(n) = g(-1)^n/n$ is in fact the example of Damanik-Killip-Simon mentioned above: The eigenvalues approach $\pm 2$ exponentially fast [4, Theorem 1].

So it seems to make sense to make $g$ $n$-dependent and approach the threshold value $g = 1$ more cautiously. The aim of this paper is to show that this idea works. Of course, since there are no positive results beyond the Damanik-Killip theorem at this point, the question of what the fastest possible decay of the $d_n$'s is must remain open. The fact that the type of counterexample used here feels right together with some experimentation with the $V^2/4$ trick from [2] have in fact led me to believe that the rate $d_n \lesssim e^{-cn^2}$ might already be the correct answer, but this is probably too bold a claim.

The plan of this paper is as follows: We will prove the estimate on the eigenvalues (part (2) of Theorem 1.1) in Sect. 2–4. We will use oscillation theory: Roughly speaking, it is possible to locate eigenvalues by counting zeros. Our basic strategy is in part inspired by the treatment of [7]. From a more technical point of view, the statement we formulate as Lemma 3.1 is very much at the heart of the matter. Part (1) of Theorem 1.1 will be proved in Sect. 5. We will use a discrete version of Levinson's Theorem (compare [6, Theorem 1.3.1]) as our main tool.

## 2. Variation of constants

We will write $V(n) = V_0(n) + V_1(n)$ with $V_0(n) = (-1)^n/n$ and $E = 2 + \epsilon$ and view $V_1$ as well as $\epsilon$ as perturbations. This strategy seems especially appropriate here because one can in fact solve (1.1) with $V = V_0$ and $E = 2$ (almost) explicitly. This observation, which is crucial for what follows, is from [2]. There is actually no need to discuss the equation for a general $E \geq 2$ here. Rather, it suffices to have good control on the solutions for $E = 2$ because we can then refer to oscillation theory at a later stage of the proof.

Let us begin with the unperturbed problem: So, consider (1.1) with $V(n) = V_0(n) = (-1)^n/n$ and $E = 2$. As observed in [2], if we define

$$\varphi_{2n} = \varphi_{2n+1} = \prod_{j=1}^{n} \left(1 + \frac{1}{2j-1}\right), \tag{2.1}$$

then $\varphi_n$ solves this equation. We find a second, linearly independent solution $\psi$ to the same equation by using constancy of the Wronskian,

$$\varphi_n \psi_{n+1} - \varphi_{n+1}\psi_n = 1, \tag{2.2}$$

and making the ansatz $\psi_n = C_n \varphi_n$. Plugging this into (2.2), we see that $\psi$ will solve (1.1) if

$$C_{n+1} - C_n = \frac{1}{\varphi_n \varphi_{n+1}}. \tag{2.3}$$

For later use, we record asymptotic formulae for these solutions. A warning may be in order here: What we call $\varphi$ in Lemma 2.1 below differs from the $\varphi$ defined in (2.1) by a constant factor. By the same token, the asymptotic formula for $C_n$ of course implies a particular choice of the constant in the general solution of (2.3).

**Lemma 2.1.** *There exist solutions $\varphi$, $\psi_n = C_n\varphi_n$ to* (1.1) *with $V = V_0$ and $E = 2$ satisfying the following asymptotic formulae:*

$$\varphi_{2n} = \varphi_{2n+1} = (2n)^{1/2} + O(n^{-1/2}), \quad C_n = \ln n + O(1/n)$$

*Sketch of proof.* Take logarithms in (2.1) and asymptotically evaluate the resulting sum by using Taylor expansions and approximating sums by integrals. Then use this information to analyze (2.3). □

To analyze the full equation, with $V$ given by (1.2), we use variation of constants. So write $T_0(n)$ for the transfer matrix of the unperturbed problem, that is,

$$T_0(n) = \begin{pmatrix} \varphi_n & \psi_n \\ \varphi_{n+1} & \psi_{n+1} \end{pmatrix}.$$

Then $\det T_0(n) = 1$ because of (2.2), and

$$T_0^{-1}(n) = \begin{pmatrix} \psi_{n+1} & -\psi_n \\ -\varphi_{n+1} & \varphi_n \end{pmatrix}.$$

It will also be convenient to write $V(n) = V_0(n) + (-1)^n W_n$, so

$$W_n = \frac{2}{n \ln n}. \tag{2.4}$$

Now let $y$ be a solution of the Schrödinger equation (1.1) with $E=2$ and potential (1.2), and introduce $D_n \in \mathbb{C}^2$ by writing

$$\begin{pmatrix} y_n \\ y_{n+1} \end{pmatrix} = T_0(n) D_n.$$

A calculation then shows that $D_n$ solves

$$D_n - D_{n-1} = A_n D_{n-1}, \qquad A_n \equiv (-1)^n W_n \varphi_n^2 \begin{pmatrix} C_n & C_n^2 \\ -1 & -C_n \end{pmatrix}. \tag{2.5}$$

We have used the fact that $\psi_n = C_n \varphi_n$.

We notice that $A_{n+1} \approx -A_n$, so we expect at least partial cancellations. To exploit this, we perform *two* steps (rather than just one) in the iteration (2.5). Clearly, $D_{2n+1} = (1+A_{2n+1})(1+A_{2n})D_{2n-1}$, so we define

$$M_n = (1+A_{2n+1})(1+A_{2n}).$$

Using the formulae $\varphi_{2n+1} = \varphi_{2n}$ and $C_{2n+1} = C_{2n} + \varphi_{2n}^{-2}$, we find that

$$\begin{aligned} M_n = 1 + (W_{2n} - W_{2n+1} + W_{2n}W_{2n+1})\varphi_{2n}^2 \begin{pmatrix} C_{2n} & C_{2n}^2 \\ -1 & -C_{2n} \end{pmatrix} \\ - W_{2n+1} \begin{pmatrix} 1 & 2C_{2n} + \varphi_{2n}^{-2} \\ 0 & -1 \end{pmatrix} + W_{2n}W_{2n+1} \begin{pmatrix} 1 & C_{2n} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

We see at this point already that the idea of doing two steps at once was a major success because this new matrix $M_n$ differs from the unity matrix by a correction of order $O(\ln n/n)$ whereas $A_n$ itself only satisfies $\|A_n\| = O(\ln n)$. For the following calculations, it will be convenient to introduce some abbreviations and write $M_n$ in the form

$$M_n = \begin{pmatrix} 1 + \epsilon_n - w_n + \rho_n & c_n(\epsilon_n - 2w_n + \rho_n - \rho_n') \\ -\epsilon_n/c_n & 1 - \epsilon_n + w_n \end{pmatrix}, \tag{2.6}$$

where

$$\epsilon_n = (W_{2n} - W_{2n+1} + W_{2n}W_{2n+1})\varphi_{2n}^2 C_{2n}, \qquad w_n = W_{2n+1}, \tag{2.7}$$

$$c_n = C_{2n}, \qquad \rho_n = W_{2n}W_{2n+1}, \qquad \rho_n' = \frac{W_{2n+1}}{\varphi_{2n}^2 C_{2n}}. \tag{2.8}$$

## 3. Counting zeros

We now want to use the difference equations from the preceding section to derive upper bounds on the number of zeros (more precisely: sign changes) of the solution $y$ on large intervals. Our goal in this section is to prove the following:

**Lemma 3.1.** *There exist $n_0 \in \mathbb{N}$ and $A > 0$ so that the following holds: If $N_1, N_2 \in \mathbb{N}$ with $N_2 \ge N_1 \ge n_0$ and*

$$\ln N_2 \le \ln N_1 + A \ln^{1/2} N_1,$$

*then there exists a solution $y$ of* (1.1) *with $E = 2$ and potential* (1.2) *satisfying $y_n > 0$ for $N_1 \le n \le N_2$.*

We start out by finding the eigenvalues and eigenvectors of $M_n$ from (2.6). The eigenvalues are given by

$$\lambda_\pm(n) = 1 + \frac{\rho_n}{2} \pm w_n \sqrt{1 + \frac{\epsilon_n \rho_n'}{w_n^2} - \frac{\rho_n}{w_n} + \frac{\rho_n^2}{4w_n^2}}.$$

We will need information on the asymptotic behavior. From Lemma 2.1 and the definitions (see (2.4), (2.7), and (2.8)), we obtain that

$$\epsilon_n = \frac{1}{n} + O\left(\frac{1}{n \ln n}\right), \qquad \rho_n, \rho_n' = O\left(\frac{1}{n^2 \ln^2 n}\right). \tag{3.1}$$

This shows, first of all, that

$$\lambda_\pm(n) = 1 \pm \frac{1}{n \ln 2n} + O\left(\frac{1}{n^2 \ln n}\right). \tag{3.2}$$

Next, write the eigenvector corresponding to $\lambda_+$ in the form $v_+ = \left(\begin{smallmatrix} 1 \\ -a_n \end{smallmatrix}\right)$. It then follows from (2.6) that $a_n$ satisfies

$$-\frac{\epsilon_n}{c_n} + (1 - \epsilon_n + w_n - \lambda_+(n))(-a_n) = 0,$$

and by using the slightly more precise formula

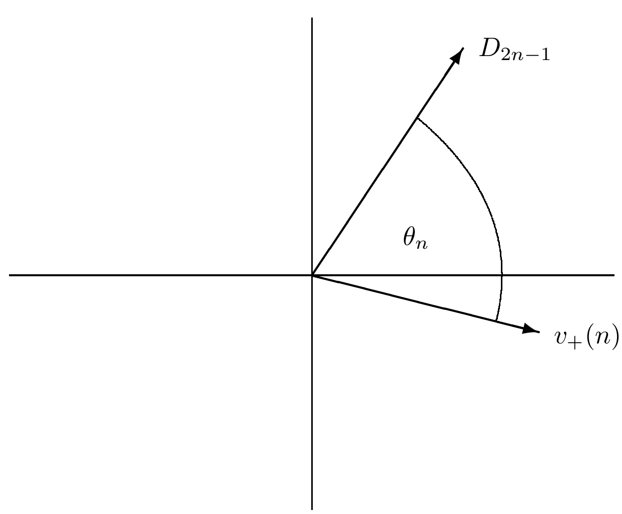
$$\lambda_+(n) = 1 + w_n + \frac{\epsilon_n \rho_n'}{2 w_n} + O\left(\frac{1}{n^2 \ln^2 n}\right)$$

instead of (3.2), we obtain from this that

$$a_n = \frac{1}{c_n} - \frac{1}{4n \ln^2 n} + O\left(\frac{1}{n \ln^3 n}\right). \tag{3.3}$$

We are interested in the sign of $y_n$, so obviously only the *direction* of $D_n$ matters. Let $\theta_n$ be the angle that $D_{2n-1}$ makes with the eigenvector $v_+$; see also Figure 1 below.

Figure 1



**Lemma 3.2.** *There exists $n_0 \in \mathbb{N}$ such that the following holds: If $n \geq n_0$ and $0 \leq \theta_n \leq \pi/2$, then $y_{2n-1} > 0$ and $y_{2n} > 0$.*

*Proof.* The condition on $\theta_n$ implies that we can write $D_{2n-1} = k_n \left(\begin{smallmatrix} 1 \\ -d_n \end{smallmatrix}\right)$ with $k_n > 0$ and $d_n \leq a_n$. Now

$$\begin{aligned} \begin{pmatrix} y_{2n-1} \\ y_{2n} \end{pmatrix} &= T_0(2n-1)D_{2n-1} = k_n \begin{pmatrix} \varphi_{2n-2} & \varphi_{2n-2}C_{2n-1} \\ \varphi_{2n} & \varphi_{2n}C_{2n} \end{pmatrix} \begin{pmatrix} 1 \\ -d_n \end{pmatrix} \\ &= k_n \begin{pmatrix} \varphi_{2n-2}(1 - C_{2n-1}d_n) \\ \varphi_{2n}(1 - C_{2n}d_n) \end{pmatrix} \end{aligned}$$

Since $k_n, \varphi_{2n-2}, \varphi_{2n} > 0$ and $C_{2n} > C_{2n-1} > 0$ (we may have to take $n$ sufficiently large here), we see that $y_{2n-1}, y_{2n}$ will certainly be positive if $1 - C_{2n}d_n = 1 - c_n d_n > 0$ or, equivalently, $1/c_n > d_n$. But (3.3) shows that $1/c_n > a_n$ for large $n$, and, as noted above, $d_n \leq a_n$, so this condition holds. $\square$

To motivate the subsequent arguments, we now make some preliminary, informal remarks about the dynamics of the recursion $D_{2n+1} = M_n D_{2n-1}$: First of all, a calculation shows that the eigenvector $v_- = v_-(n)$ associated with the small eigenvalue $\lambda_- < 1$ is of the form $v_- = \left(\begin{smallmatrix} 1 \\ -b_n \end{smallmatrix}\right)$ with $b_n > a_n$, so $v_-$ lies below $v_+$. However, $b_n - a_n = O(\ln^{-2} n)$, so $v_+$ and $v_-$ are almost parallel.

Now an application of a $2 \times 2$ matrix moves the vector towards the eigenvector corresponding to the large eigenvalue. So in the case at hand, $D$ will approach $v_+$, or, in other words, $\theta_n$ will decrease. At the same time, $v_+(n)$ approaches the positive $x$-axis, but this is a comparatively small effect. Nevertheless, a crossing between $D$ and $v_+$ will eventually occur. Our task is to bound from below the number of iterations it takes (starting from $\theta = \pi/4$, say) to reach this crossing.

We will use the eigenvector $v_+ = v_+(n) = \left(\begin{smallmatrix} 1 \\ -a_n \end{smallmatrix}\right)$ and the orthogonal vector $\left(\begin{smallmatrix} a_n \\ 1 \end{smallmatrix}\right)$ as our basis of $\mathbb{R}^2$. As $\theta_n$ was defined as the angle between $D_{2n-1}$ and $v_+(n)$, it follows that $D_{2n-1}$ is a constant multiple of the vector

$$\cos\theta_n v_+(n) + \sin\theta_n \begin{pmatrix} a_n \\ 1 \end{pmatrix}. \tag{3.4}$$

Conversely, we can find $\theta$ using the fact that $D$ has such a representation. More precisely, to compute $\theta_{n+1}$ from $\theta_n$, we apply the matrix $M_n$ to the vector from (3.4) and then take scalar products with $v_+(n+1)$ and $\left(\begin{smallmatrix} a_{n+1} \\ 1 \end{smallmatrix}\right)$. These operations produce multiples of $\cos\theta_{n+1}$ and $\sin\theta_{n+1}$, respectively. We omit the details of this routine calculation. The result is as follows: If we introduce $t_n = \tan\theta_n$, then

$$t_{n+1} = \frac{s_n t_n + \lambda_+(n)(a_{n+1} - a_n)}{\widetilde{s}_n t_n + \lambda_+(n)(1 + a_n a_{n+1})}, \tag{3.5}$$

where $a_n$ was defined above (see also (3.3)) and

$$\begin{aligned} s_n &= (a_{n+1}, 1) M_n \begin{pmatrix} a_n \\ 1 \end{pmatrix}, \\ \widetilde{s}_n &= (1, -a_{n+1}) M_n \begin{pmatrix} a_n \\ 1 \end{pmatrix}. \end{aligned}$$

From (2.6), (3.1), (3.3), and Lemma 2.1, we obtain the asymptotic formulae

$$s_n = 1 + a_n a_{n+1} + O\left(\frac{1}{n \ln n}\right), \qquad \widetilde{s}_n = \frac{\ln n}{n} + O\left(\frac{1}{n}\right). \tag{3.6}$$

We will prove Lemma 3.1 by analyzing the recursion (3.5). As a preliminary, we observe the following:

**Lemma 3.3.** *There exists $n_0 \in \mathbb{N}$ so that the following holds: If $n \geq n_0$ and $0 \leq t_n \leq 1/\ln n$, then also $t_{n+1} \leq 1/\ln(n+1)$.*

*Proof.* Let

$$f(x) = \frac{s_n x + \lambda_+(a_{n+1} - a_n)}{\widetilde{s}_n x + \lambda_+(1 + a_n a_{n+1})}$$

be the function from (3.5). Then

$$f'(x) = \frac{\lambda_+}{(\widetilde{s}_n x + \lambda_+(1 + a_n a_{n+1}))^2} \left( s_n(1 + a_n a_{n+1}) - \widetilde{s}_n(a_{n+1} - a_n) \right),$$

and since $a_{n+1} - a_n = O(1/(n \ln^2 n))$, the derivative is positive for large enough $n$. Therefore, $t_{n+1} = f(t_n) \leq f(1/\ln n)$, and by dividing through by $1 + a_n a_{n+1}$ and using (3.2), (3.3), and (3.6), we see that

$$t_{n+1} \leq \frac{\left(1 + O\left(\frac{1}{n \ln n}\right)\right) \frac{1}{\ln n} + O\left(\frac{1}{n \ln^2 n}\right)}{\frac{1}{n} + 1 + O\left(\frac{1}{n \ln n}\right)} = \left(1 - \frac{1}{n}\right) \frac{1}{\ln n} + O\left(\frac{1}{n \ln^2 n}\right).$$

On the other hand,

$$\ln(n+1) = \ln n + \ln\left(1 + \frac{1}{n}\right) = \ln n + O\left(\frac{1}{n}\right),$$

so the claim follows. □

We are now ready for the *proof of Lemma 3.1.* We must show that when solving the basic recursion $D_{2n+1} = M_n D_{2n-1}$ (or its variant (3.5)), at least as much time as specified in the statement is spent in the region where $y_n > 0$. We will in fact show that $t_n$ spends such an amount of time already in the region where

$$\ln^{-3/2} n \leq t_n \leq \ln^{-1} n. \tag{3.7}$$

Lemma 3.2 shows that this condition indeed implies that $y_{2n-1}, y_{2n} > 0$. In fact, (3.7) might look unnecessarily restrictive so that the whole analysis would appear to be rather crude. However, an argument similar to the one we are about to give shows that the time spent in the neglected regions is at most of the same order of magnitude. More precisely, if $0 \leq t_n \leq \ln^{-3/2} n$ for $N_1 \leq n \leq N_2$, then these $N_1$, $N_2$ also satisfy the estimate from Lemma 3.1. Moreover, if $\ln^{-1} n \leq t_n \leq M$ for $N_1 \leq n \leq N_2$, then $N_2 \leq C N_1$, with $C$ independent of $M > 0$. These remarks, together with a more careful analysis of the crossing between $D$ and $v_+$, show that the condition from Lemma 3.1 is sharp.

Let us now proceed with the strategy outlined above. Assume that (3.7) holds. From (3.5), we obtain that

$$t_{n+1} - t_n = \frac{\left[s_n - \lambda_+(n)(1 + a_n a_{n+1})\right] t_n + \lambda_+(n)(a_{n+1} - a_n) - \widetilde{s}_n t_n^2}{\widetilde{s}_n t_n + \lambda_+(n)(1 + a_n a_{n+1})}.$$

Note that $s_n - \lambda_+(n)(1 + a_n a_{n+1}) = O(1/(n \ln n))$. Also, $t_n \leq \ln^{-1} n$ by assumption, so the first term in the numerator is of the order $O(1/(n \ln^2 n))$. As for the next term, recall that $a_{n+1} - a_n = O(1/(n \ln^2 n))$. Finally, since we are assuming that $t_n \geq \ln^{-3/2} n$, we have that $\widetilde{s}_n t_n^2 \gtrsim 1/(n \ln^2 n)$, so up to a constant factor, this term

is not smaller than the other two summands from the numerator. The denominator clearly is of the form $1 + o(1)$. So, putting things together, we see that

$$t_{n+1} - t_n \geq -C\frac{\ln n}{n} t_n^2,$$

with $C > 0$. It will in fact be convenient to write this in the form

$$t_{n+1} - t_n \geq -C\frac{\ln n}{n} t_n t_{n+1},$$

with an adjusted constant $C > 0$. We can then introduce $r_n = 1/t_n$, and this new variable obeys

$$r_{n+1} - r_n \leq C\frac{\ln n}{n}. \tag{3.8}$$

This was derived under the assumptions that $n$ is sufficiently large and that we have the two bounds

$$\ln n \leq r_n \leq \ln^{3/2} n.$$

Our final task is to use (3.8) to find an estimate on the first $n$ for which the second inequality fails to hold. Recall also that Lemma 3.3 says that there can't be any such problems with the first inequality. Suppose that $N_1 \in \mathbb{N}$ is sufficiently large and $r_{N_1} = \ln N_1$. We can proceed by induction: What we have just shown says that if $r_j \leq \ln^{3/2} j$ for $j = N_1, N_1 + 1, \ldots, n-1$, then $r_n \geq \ln n$ and

$$r_n \leq \ln N_1 + C \sum_{j=N_1}^{n-1} \frac{\ln j}{j}. \tag{3.9}$$

So we can keep going as long as the right-hand side of (3.9) is $\leq \ln^{3/2} n$. Now clearly

$$\sum_{j=N_1}^{n-1} \frac{\ln j}{j} \lesssim \ln^2 n - \ln^2 N_1,$$

so, recalling from the statement of Lemma 3.1 what we are actually trying to prove, we see that it suffices to show that given a constant $C > 0$, we can find $A > 0$, $n_0 \in \mathbb{N}$ so that if $N_1 \geq n_0$, then the condition

$$\ln N_2 \leq \ln N_1 + A \ln^{1/2} N_1 \tag{3.10}$$

implies that

$$\ln N_1 + C(\ln^2 N_2 - \ln^2 N_1) \leq \ln^{3/2} N_2. \tag{3.11}$$

Indeed, it will then follow that $y_n > 0$ for $n = 2N_1 - 1, \ldots, 2N_2$, and a simple adjustment gives Lemma 3.1 as originally stated, without the factors of 2.

But the above claim is actually quite obvious: By taking squares in (3.10), we obtain the estimate

$$\ln^2 N_2 - \ln^2 N_1 \leq 2A \ln^{3/2} N_1 + A^2 \ln N_1 \leq 3A \ln^{3/2} N_1$$

(say), and if also $3AC < 1$, then (3.11) follows at once. $\square$

## 4. Oscillation theory

In this section, we will use Lemma 3.1 to derive the desired estimate on the discrete eigenvalues. For the time being, we are concerned with eigenvalues $E_n > 2$; in particular, we then have that $d_n = E_n - 2$. We will need some standard facts from oscillation theory; for proofs, we refer the reader to [9, Chapter 4]. This reference gives a careful discussion of all the results we will need (and several others), but some caution is required when looking up results in [9] because the operators discussed there are the negatives of the operators generated by (1.1).

We first state a couple of comparison theorems: If $u_1$, $u_2 \not\equiv 0$ both solve the same Schrödinger equation (1.1), then the number of zeros (more precisely: sign changes) on any fixed interval differs by at most one [9, Lemma 4.4]. Also, if $E \leq E'$ and $u$, $u'$ solve the Schrödinger equation with energies $E$ and $E'$, respectively, and these solutions have the same initial phase at $N_1$ (i.e. $u(N_1+1)/u(N_1) = u'(N_1+1)/u'(N_1)$), then for any $N_2 > N_1$, $u$ has at least as many zeros on $\{N_1, \ldots, N_2\}$ as $u'$. This can be deduced from [9, Theorem 4.7].

Finally, the following is a more direct consequence of [9, Theorem 4.7]: If $u$ solves (1.1), $u(0) = 0$, $u(1) = 1$, and $u$ has $N$ zeros on $\mathbb{N}$, then $E_N \leq E < E_{N-1}$. Here, we of course implicitly assume that the eigenvalues $E_n$ are arranged in their natural order: $E_0 > E_1 > E_2 > \ldots$ (and $E_{-1} := \infty$). In other words, it is possible to locate discrete eigenvalues by counting zeros.

Let us now use these facts to derive Theorem 1.1(2) from Lemma 3.1. Define $N_n = \exp(\gamma n^2) + x_n$, with $0 < \gamma < A^2/4$, where $A$ is the constant from Lemma 3.1; $x_n \in [0, 1)$ is chosen so that $N_n \in \mathbb{N}$. Then $\ln N_{n+1} \leq \ln N_n + A \ln^{1/2} N_n$ for all sufficiently large $n$, so Lemma 3.1 and the facts reviewed in the preceding paragraphs imply that any nontrivial solution $u$ of (1.1) with $E = 2$ has at most $C+n$ zeros on $\{1, \ldots, N_n\}$, where $C$ is a fixed constant, independent of $n$. Moreover, the same holds for any nontrivial solution of (1.1) with $E \geq 2$ because increasing the energy leads to fewer zeros by the comparison theorem quoted above.

Now fix $E > 2$ and let $d = E - 2$; for convenience, also assume that $d < 1/2$, say. Then if $m \geq 3/d$, then $|V(m')| < d$ for all $m' \geq m$, and thus the operator on $\ell_2(\{m, m+1, \ldots\})$ doesn't have any spectrum in $[2+d, \infty)$. As a consequence, any solution $u$ of the Schrödinger equation (1.1) with $E = 2 + d$ has at most one zero on $\{m, m+1, \ldots\}$. If this is combined with what has been observed above, it follows that an arbitrary non-trivial solution $u$ to (1.1) with $E = 2 + d$ has at most $C + n(d)$ zeros on $\mathbb{N}$; here $n(d)$ must be chosen so that $N_{n(d)} \geq 3/d$. In other words, it is possible to pick $n(d) \lesssim \ln^{1/2} d^{-1}$.

To sum this up: Any non-trivial solution $u$ to (1.1) with $E = 2 + d$ has at most $C_1 + C_2 \ln^{1/2} d^{-1} \leq C_3 \ln^{1/2} d^{-1}$ zeros on $\mathbb{N}$. We can now use that part of oscillation theory that relates the location of the eigenvalues to the number of zeros. It follows that if $N = [C \ln^{1/2} d^{-1}]$, then the $N$th eigenvalue, $E_N$, satisfies $E_N \leq 2 + d$. By rearranging, we see that $E_N - 2 \leq \exp(-cN^2)$, as claimed in part (2) of Theorem 1.1.

Of course, we haven't talked about eigenvalues $< -2$ yet, but this part of the claim is established by a completely analogous analysis. We can use the fact that $-E$ is an eigenvalue of the original problem if and only if $E$ is an eigenvalue for the potential $-V$. This reduces matters to the discussion of the eigenvalues bigger than 2, but for the two potentials $V$ and $-V$. We have just discussed $V$, and, not

surprisingly, it turns out that the sign change is quite irrelevant and we can simply run the whole argument again, with only a few very minor modifications. So we will not give any details, and these general remarks conclude the proof of part (2) of Theorem 1.1.

## 5. Asymptotic integration

What we will do below is modelled on the treatment of similar problems in the continuous setting. See in particular [6, Sect. 4.3].

We want to analyze the solutions of the discrete Schrödinger equation (1.1) with potential (1.2) and $E = 0$. For the following computations, it will be convenient to write $V(n) = (-1)^n 2v_n$, so

$$v_n = \frac{1}{2n}\left(1 + \frac{2}{\ln n}\right)$$

for $n \geq 3$. Given a solution $y$ of

$$y_{n+1} + y_{n-1} + (-1)^n 2v_n y_n = 0,$$

introduce $Y_n = \left(\begin{smallmatrix} y_{n-1} \\ y_n \end{smallmatrix}\right)$. Then $Y$ solves

$$Y_{n+1} = \begin{pmatrix} 0 & 1 \\ -1 & (-1)^{n+1} 2v_n \end{pmatrix} Y_n. \tag{5.1}$$

We again use a variation of constants type transformation, treating $V$ as the perturbation. So define a new variable $Z$ by $Y_n = T_n Z_n$, where

$$T_n = \begin{pmatrix} \cos \pi(n-1)/2 & \sin \pi(n-1)/2 \\ \cos \pi n/2 & \sin \pi n/2 \end{pmatrix};$$

note that indeed $T_{n+1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) T_n$, that is, $T$ solves the unperturbed equation. A calculation shows that $Z$ obeys

$$Z_{n+1} = Z_n - v_n \begin{pmatrix} 0 & 1 + (-1)^{n+1} \\ 1 + (-1)^n & 0 \end{pmatrix} Z_n.$$

Here, we have used the fact that the trigonometric functions only take the values $0, \pm 1$ at integer multiples of $\pi/2$, and, for example, $\cos^2 n\pi/2 = (1 + (-1)^n)/2$.

Next, we diagonalize the non-oscillating part of the perturbation: To this end, write

$$Z_n = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} W_n.$$

Then $W$ solves

$$W_{n+1} = W_n + v_n \begin{pmatrix} -1 & (-1)^{n+1} \\ (-1)^n & 1 \end{pmatrix} W_n.$$

Finally, we can now approximately get rid of the oscillating part with the help of a transformation of the type

$$W_n = (1 + v_n A_n) U_n.$$

The matrix $A_n$ will be chosen shortly; it will satisfy $A_n = O(1)$. Since $v_n^2, v_{n+1} - v_n \in \ell_1$, we then have that

$$(1 + v_{n+1} A_{n+1})^{-1} = 1 - v_n A_{n+1} + B_n$$

with $B_n \in \ell_1$. It thus follows that

$$U_{n+1} = U_n + v_n \left[ A_n - A_{n+1} + \begin{pmatrix} -1 & (-1)^{n+1} \\ (-1)^n & 1 \end{pmatrix} \right] U_n + R_n U_n,$$

with $R_n \in \ell_1$. This suggests that we take

$$A_n = \frac{(-1)^n}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and this choice leads to the equation

$$U_{n+1} = \begin{pmatrix} 1 - v_n & 0 \\ 0 & 1 + v_n \end{pmatrix} U_n + R_n U_n \tag{5.2}$$

for $U$.

We expect that the summable perturbation $R$ does not change the asymptotics of the solutions. We are especially interested in the decaying solution, and we want to show that, more specifically, there exists a solution $U$ satisfying $U_n \approx \left( \prod_{j=j_0}^{n-1} (1 - v_j) \right) \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$. To do this, we mimic the proof of Levinson's Theorem. We will basically follow the presentation given in [1] (but see also [6, Sect. 1.4]). When appropriate, we will right away specialize to the case at hand although the underlying arguments are actually of a much more general character throughout.

Consider the "integral equation"

$$C_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{j=n}^{\infty} \frac{1}{1 - v_j} \begin{pmatrix} 1 & 0 \\ 0 & \prod_{k=n}^{j} \frac{1-v_k}{1+v_k} \end{pmatrix} R_j C_j \equiv e_1 - (TC)_n \tag{5.3}$$

We will see later that this is basically a way of rewriting (5.2). Since $R \in \ell_1$, $0 < v_n \leq c < 1$, and $v_n \to 0$, the sum from (5.3) defines a bounded operator on $\ell_\infty(\{j_0, j_0 + 1, \ldots\}; \mathbb{C}^2)$ (bounded sequences on $\{n : n \geq j_0\}$ taking values in $\mathbb{C}^2$). More precisely,

$$\|TC\|_\infty \leq \frac{1}{1 - v_{j_0}} \sum_{j=j_0}^{\infty} |R_j| \cdot \|C\|_\infty.$$

So if we take $j_0$ sufficiently large, then in fact $\|T\| < 1$ and thus $1 + T$ is boundedly invertible on $\ell_\infty(\{j_0, j_0 + 1, \ldots\}; \mathbb{C}^2)$. In particular, $C \equiv (1 + T)^{-1} e_1$ is a bounded solution to (5.3). This boundedness of course also makes sure that the series from (5.3) converges.

Moreover, it follows from (5.3) that $C_n \to e_1$ as $n \to \infty$. Finally, as already announced, we obtain a solution to the original equation (5.2) from this $C$: define

$$U_n = \left[ \prod_{j=j_0}^{n-1} (1 - v_j) \right] C_n \equiv p_n C_n. \tag{5.4}$$

Then $U_n$ solves (5.2) for $n \geq j_0$. To verify this claim, call the diagonal matrix from (5.2) $\Lambda_n$, so that (5.2) becomes $U_{n+1} = (\Lambda_n + R_n) U_n$. Next observe that $p_{n+1} e_1 = p_n \Lambda_n e_1$ and

$$p_{n+1} \begin{pmatrix} 1 & 0 \\ 0 & \prod_{k=n+1}^{j} \frac{1-v_k}{1+v_k} \end{pmatrix} = p_n \Lambda_n \begin{pmatrix} 1 & 0 \\ 0 & \prod_{k=n}^{j} \frac{1-v_k}{1+v_k} \end{pmatrix}.$$

So if $U_n$ is defined by (5.4) and $C_n \in \ell_\infty$ solves (5.3), then

$$U_n = p_n e_1 - p_n \sum_{j=n}^{\infty} \frac{1}{1-v_j} \begin{pmatrix} 1 & 0 \\ 0 & \prod_{k=n}^{j} \frac{1-v_k}{1+v_k} \end{pmatrix} R_j C_j,$$

thus

$$\begin{aligned}
U_{n+1} &= p_{n+1} e_1 - p_{n+1} \sum_{j=n+1}^{\infty} \frac{1}{1-v_j} \begin{pmatrix} 1 & 0 \\ 0 & \prod_{k=n+1}^{j} \frac{1-v_k}{1+v_k} \end{pmatrix} R_j C_j \\
&= \Lambda_n p_n e_1 - p_n \Lambda_n \sum_{j=n+1}^{\infty} \frac{1}{1-v_j} \begin{pmatrix} 1 & 0 \\ 0 & \prod_{k=n}^{j} \frac{1-v_k}{1+v_k} \end{pmatrix} R_j C_j \\
&= \Lambda_n p_n e_1 - p_n \Lambda_n \sum_{j=n}^{\infty} \frac{1}{1-v_j} \begin{pmatrix} 1 & 0 \\ 0 & \prod_{k=n}^{j} \frac{1-v_k}{1+v_k} \end{pmatrix} R_j C_j \\
&\quad + p_n \Lambda_n \frac{1}{1-v_n} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1-v_n}{1+v_n} \end{pmatrix} R_n C_n \\
&= \Lambda_n U_n + p_n R_n C_n = (\Lambda_n + R_n) U_n,
\end{aligned}$$

as required.

We can now go back to the original variable $Y$:

$$Y_n = T_n \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (1 + v_n A_n) U_n$$

See also (5.1). Since $U_n = p_n(e_1 + o(1))$ as $n \to \infty$, it follows that there is a solution $y$ of the Schrödinger equation satisfying

$$|y_n| = (1 + o(1)) \prod_{j=j_0}^{n} (1 - v_j) \qquad (n \to \infty).$$

Now

$$\begin{aligned}
\ln \prod_{j=j_0}^{n} (1 - v_j) &= \sum_{j=j_0}^{n} \ln(1 - v_j) = - \sum_{j=j_0}^{n} v_j + O(1) \\
&= - \sum_{j=j_0}^{n} \left( \frac{1}{2j} + \frac{1}{j \ln j} \right) + O(1) = -\frac{1}{2} \ln n - \ln \ln n + O(1),
\end{aligned}$$

so

$$y_n^2 \lesssim \frac{1}{n \ln^2 n},$$

and $y$ is indeed square summable. The proof of Theorem 1.1 is complete. □

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